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A Non-linear Interpolation Formula

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In 1915, E. T. Whittaker showed that an L^2 function, whose Fourier tranform has compact support, can be interpolated on an arithmetic progression: the well-known *Cardinal series*, or *Sampling Representation*. In the present paper, a class of non-linear interpolation formulae like the sampling representation are derived, in which the points of interpolation consist, in part, of the eigenvalues of certain differential operators acting on a Riemann surface of genus, $g > 1$.

In 1915, Whittaker [3] showed that an L^2 function, whose Fourier transform has compact support, can be interpolated on an arithmetic progression; e.g., if $\hat{g} \in L^2(-\pi, \pi)$, then

$$g(x) = \frac{\sin \pi x}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n g(n)}{x-n}. \tag{1}$$

Whittaker referred to (1) as the *Cardinal series*. Later, (1) gained widespread use in information theory and came to be known as the *sampling representation*. In the present paper a class of similar interpolation formulae is shown to hold, where the points of interpolation consist, in part, of numbers related to the eigenvalues of certain differential operators. These formulae will be deduced from a modified version of the Selberg Trace Formula ([2, pp. 72–74]; see also [1, pp. 451–457]).

We shall need some notation and terminology, which agrees, with minor exceptions, with that in [2]. Thus, let M denote a Riemann surface of genus $g > 1$, and let G be a uniformizing, discrete subgroup of $S1(2, \mathbb{R})$, with $M = G\backslash H$. The elements of G may be divided into four types: the identity, and the hyperbolic, elliptic, and parabolic elements. It will be assumed that M is compact, in which case there are no parabolic elements. In order to gain a further economy in our formulae, it will be assumed that G contains no elliptic elements; i.e., G is strictly hyperbolic. The area of M will be denoted by $A(M)$.

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Let $\{P\}$ denote the distinct, hyperbolic conjugacy classes of G; and for each hyperbolic element P, let P_0 be the primitive element for which $P = P_0^k$, $k \geqslant 1$. The symbols $N(P)$ and $N(P_0)$ will denote the norms of P and P_0, respectively (and these satisfy $N(P) > 1$). It will be important later to distinguish the least such norm, $N_1 > 1$,

$$N_1 = \min_{N(P) > 1} N(P).$$

Consider the differential operator $Df = y^2 \Delta f$, on M, where $f(z) = f(x + iy)$. A complete set of orthonormal solutions of $Df + \lambda f = 0$ forms a basis of $L^2(M)$. We list the corresponding eigenvalues in increasing order: $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \to \infty$. And we put $\lambda_n = \frac{1}{4} + r_n^2$.

Finally, the Fourier transform of $\varphi \in L^2(\mathbb{R})$ is denoted by

$$\hat{\varphi}(t) = \int_{-\infty}^{\infty} e^{2\pi i t x} \varphi(x)\, dx,$$

which differs, by a factor 2π, from the terminology of [1, 2].

LEMMA 1 (Selberg). *Let $\hat{\varphi}(w) = \hat{\varphi}(u + iv)$ denoted any analytic function on $|v| \leqslant 1/4\pi + \delta$ such that $\hat{\varphi}(-w) = \hat{\varphi}(w)$ and $|\hat{\varphi}(w)| \leqslant C(1 + |w|)^{-2-\delta}$ for positive constants C and δ. Then*

$$\sum_{r_n} \left(\frac{r_n}{2\pi}\right) = \frac{A(M)}{2\pi} \int_{-\infty}^{\infty} u\hat{\varphi}\left(\frac{u}{2\pi}\right) \tanh \pi u\, du + 2 \sum_{\{P\}} C(P)\, \varphi(\ln N(P)), \quad (2)$$

where the latter sum is over the distinct, hyperbolic conjugacy classes of G, and where

$$C(P) = \frac{\ln N(P_0)}{N(P)^{1/2} - N(P)^{-1/2}}.$$

Proof. See [2]. Note that both solutions of $\lambda_n = \frac{1}{4} + r_n^2$ are counted on the left side of (2); and if $\lambda_n = \frac{1}{4}$, then $r_n = 0$ is counted twice. ∎

It will be necessary to have a modified version of (2) in which $\hat{\varphi}$ need not be even.

LEMMA 2. *Let $\hat{\varphi}(w)$ satisfy the hypotheses of Lemma 1, except that $\hat{\varphi}$ need not be even. Then*

$$\sum_{r_n} \hat{\varphi}\left(\frac{r_n}{2\pi}\right) = \sum_{\{P\}} C(P)[\varphi(\ln N(P)) + \varphi(-\ln N(P))]$$
$$+ \frac{A(M)}{2\pi} \int_0^{\infty} \left[\varphi(0) - \frac{\varphi(u) + \varphi(-u)}{2}\right] \frac{\coth(u/2)}{\sinh(u/2)}\, du. \quad (3)$$

Proof. Formula (3) will follow directly from (2). We first show that

$$\int_{-\infty}^{\infty} u\hat{\varphi}\left(\frac{u}{2\pi}\right)\tanh \pi u\, du = \int_0^{\infty} [\varphi(0)-\varphi(u)]\frac{\coth(u/2)}{\sinh(u/2)}\, du,$$

provided φ satisfies the hypotheses of Lemma 1. To see this, substitute

$$\tanh \pi u = \frac{1}{\pi}\int_0^{\infty} \frac{\sin ux}{\sinh(x/2)}\, dx$$

in the integral in (2). The resulting integral is absolutely convergent; and, since φ is even, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} u\hat{\varphi}\left(\frac{u}{2\pi}\right)\tanh \pi u\, du &= \frac{1}{\pi}\int_0^{\infty}\frac{dx}{\sinh(x/2)}\int_{-\infty}^{\infty} u\hat{\varphi}\left(\frac{u}{2\pi}\right)\sin ux\, du \\
&= 4\pi\int_0^{\infty}\frac{dx}{\sinh(x/2)}\int_{-\infty}^{\infty} t\hat{\varphi}(t)\sin 2\pi tx\, dt \\
&= -2\int_0^{\infty}\frac{dx}{\sinh(x/2)}\frac{d}{dx}\int_{-\infty}^{\infty}\hat{\varphi}(t)\cos 2\pi tx\, dt \\
&= -2\int_0^{\infty}\frac{\varphi'(x)}{\sinh(x/2)}\, dx \\
&= \int_0^{\infty}[\varphi(0)-\varphi(x)]\frac{\coth(x/2)}{\sinh(x/2)}\, dx. \qquad (4)
\end{aligned}$$

Now let φ satisfy the hypotheses of Lemma 2, except φ need not be even. Then formula (2) is valid for $\varphi_1(u) = (\varphi(u)+\varphi(-u))/2$. Substituting (4) in (2) (noting that the r_n are symmetric about the origin) and using φ_1 in place of φ, we obtain (3). ∎

We take a fixed hyperbolic, Fuchsian group G, and let $\{r_n\}_{-\infty}^{\infty}$, P, P_0, $N(P)$, $N(P_0)$, N_1 be as described before. Let $L = \ln N_1 > 0$; and let

$$C(P) = \frac{\ln N(P_0)}{N(P)^{1/2} - N(P)^{-1/2}}.$$

Then we have

THEOREM. *Let $g \in L^2(\mathbb{R})$ be a function whose Fourier transform is twice continuously differentiable and has support $[0, L]$, where $L = \ln N_1 > 0$. Then g may be interpolated by means of the formula*

$$g\left(\frac{s}{2\pi i}\right) = \frac{N(s)}{D(s)},$$

where

$$N(s) = \sum_{r_n} \frac{g(-r_n/2\pi)}{s - ir_n} + \frac{A(M)}{2\pi} \sum_{n=0}^{\infty} \frac{2n+1}{s+n+\frac{1}{2}}\, g\left(\frac{n+\frac{1}{2}}{2\pi i}\right),$$

and

$$\begin{aligned} D(s) = s \sum_{r_n} &\left(\frac{1}{s^2 + r_n^2} - \frac{1}{4 + r_n^2}\right) \\ &- \frac{sA(M)}{\pi} \sum_{n=0}^{\infty} \left(\frac{1}{s+n+\frac{1}{2}} - \frac{1}{n+\frac{5}{2}}\right) + As, \end{aligned}$$

where $A = \frac{1}{2} \sum C(P)\, N(P)^{-2}$.

Proof. Fix $s > \frac{1}{2}$, and for each $0 < y < L$, define a function $\varphi_y(x) = \varphi_y(x, s)$ by

$$\begin{aligned} \varphi_y(x) &= (x-y)^2\, e^{-sx}, \qquad && x > y, \\ &= 0, && x \leqslant y. \end{aligned}$$

Since $s \geqslant \frac{1}{2}$, the function

$$\begin{aligned} \hat{\varphi}_y(w) &= \int_y^{\infty} (x-y)^2\, e^{(2\pi i w - s)x}\, dx \\ &= -\frac{2e^{(2\pi i w - s)y}}{(2\pi i w - s)^3} \end{aligned}$$

satisfies the conditions of Lemma 2. (Note that $\hat{\varphi}_y(w)$ is not even.) Substituting $\hat{\varphi}_y(w)$ into (3) yields

$$\sum_{r_n} \hat{\varphi}_y\left(\frac{r_n}{2\pi}\right) = \sum_{\{p\}} C(P)\, \varphi_y(\ln N(P)) - \frac{A(M)}{4\pi} \int_0^{\infty} \varphi_y(u) \frac{\coth(u/2)}{\sinh(u/2)}\, du, \quad (5)$$

as $\varphi_y(u) = 0$ for $u \leqslant 0$. Noting that

$$\frac{\coth(u/2)}{\sinh(u/2)} = 2 \sum_{n=0}^{\infty} (2n+1)\, e^{-(n+1/2)u},$$

a short computation yields (after rearranging terms)

$$\begin{aligned} &\sum C(P)(\ln N(P) - y)^2\, e^{-s \ln N(P)} \\ &\qquad = -2 \sum_{r_n} \frac{e^{(ir_n - s)y}}{(ir_n - s)^3} + \frac{A(M)}{\pi} \sum_{n=0}^{\infty} \frac{e^{-(n+1/2+s)y}}{(n+\frac{1}{2}+s)^3}. \end{aligned} \quad (6)$$

Now multiply (6) by $\hat{g}(y)\,e^{sy}$ and integrate over $0 < y < L$, to obtain

$$\frac{d^2}{ds^2}\left[g\left(-\frac{s}{2\pi i}\right)\sum C(P)\,N(P)^{-s}\right]$$
$$= -2\sum_{r_n}\frac{g(-r_n/2\pi)}{(ir_n - s)^3} + \frac{A(M)}{\pi}\sum_{n=0}^{\infty}(2n+1)\,\frac{g((n+\frac{1}{2})/2\pi i)}{(n+\frac{1}{2}+s)^3}. \qquad (7)$$

Since $\hat{g}$ is twice continuously differentiable, g satisfies $|g(w)| \ll |w|^{-2}$ as $w \to \infty$; and since

$$\sum_{0 \leqslant r_n \leqslant T} 1 \sim \frac{A(M)}{4\pi}\,T^2, \qquad T \to \infty,$$

then the series

$$\sum_{r_n}\frac{g(r_n/2\pi)}{ir_n - s}$$

and

$$\sum_{n=0}^{\infty} g\left(\frac{n+\frac{1}{2}}{2\pi i}\right)$$

are absolutely convergent. Hence, we may integrate (7) termwise from s to ∞ twice, obtaining

$$g\left(-\frac{s}{2\pi i}\right)\sum C(P)\,N(P)^{-s}$$
$$= \sum_{r_n}\frac{g(-r_n/2\pi)}{s - ir_n} + \frac{A(M)}{2\pi}\sum_{n=0}^{\infty}\frac{2n+1}{n+\frac{1}{2}+s}\,g\left(\frac{n+\frac{1}{2}}{2\pi i}\right). \qquad (8)$$

If we take $\varphi(u) = (1/2s)\,e^{-s|u|} - \frac{1}{4}e^{-2|u|}$ in Lemma 1, then

$$\hat{\varphi}\left(\frac{w}{2\pi}\right) = \frac{1}{w^2+s^2} - \frac{1}{w^2+4},$$

and (2) yields

$$\sum C(P)\,N(P)^{-s} = s\sum_{r_n}\left(\frac{1}{s^2+r_n^2} - \frac{1}{4+r_n^2}\right)$$
$$-\frac{sA(M)}{\pi}\sum_{n=0}^{\infty}\left(\frac{1}{s+n+\frac{1}{2}} - \frac{1}{n+\frac{5}{2}}\right) + As. \qquad (9)$$

Hence, $\sum C(P)\, N(P)^{-s}$ is a meromorphic function with residue 1 at $s = \pm i r_n$ (since each r_n is counted twice) and residue $(n + \frac{1}{2})\, A(M)/\pi$ at $s = -(n + \frac{1}{2})$, $n \geqslant 0$. It follows that the ratio $N(S)/D(S)$ is an entire function, which shows that (8) (which was derived for $s > \frac{1}{2}$) holds for all complex s.

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